ME599 lecture 06 - small-scale fluid flows

As scale decreases, surface/volume increases so friction increases

→ slip counteracts this

Knudsen number = quantifies importance of slip

$$Kn = \frac{\lambda}{L_0}$$ — λ: mean free path, L_0: length scale — gases

$$Kn = \frac{b}{L_0}$$ — b: slip length — liquids

e.g., L_0 = pipe diameter

start from the ideal gas law,

$$p = n k_B T$$

k_B — Boltzmann's constant, 1.38 E-23 J/K

n — # density = 2.7 E 25 /m^3

mean molecular spacing, $\delta \propto n^{-1/3} \approx 3E{-9}$ m = 3 nm

d = hard sphere diameter $\approx 10^{-10}$ m

so we define a "dilute gas" ⇒ $\delta/d \gg 1$

⇒ mostly atom-atom collisions

rather than multi-atom collisions

gas mean free path

$$\lambda = \frac{1}{\pi d^2 n \sqrt{2}} = \frac{kT}{\pi p d^2 \sqrt{2}}$$, substituting the ideal gas law

hard sphere diameter — $f(\text{pressure})$

flow regimes:

$Kn < 0.01$ = continuum (no slip)

$0.01 < Kn < 0.1$ = slip

$0.1 < Kn < 10$ = transition

$Kn > 10$ = "free molecule" ~ macroscopic property meanings break down

$\Rightarrow$ direct simulations of Boltzmann equation

example of λ: air @ 1 atm, 300 K, λ = 65 nm

0.001 atm, 300 K, λ = 65 μm !

1 atm, 1200 K, λ = 2.6 μm

$\Downarrow$

microchannel < 250 μm is in slip regime

Navier - stokes eqns for unsteady incompressible flow

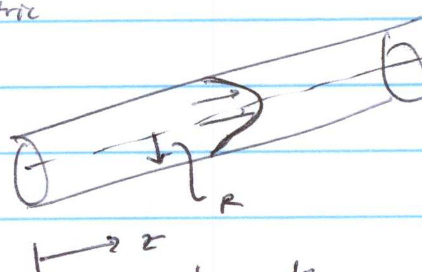
$$\frac{d\vec{V}}{dt} = \frac{-\nabla p}{\rho} + \nu^2 \nabla \vec{V}^2 + f, \quad \nu = \frac{\mu}{\rho}$$

$$\nabla \cdot \vec{V} = 0$$

$\Rightarrow$ let's quantify the effect of slip on friction in a small pipe

flow is 1-dimensional, θ-symmetric

reduced N/S eqn.



$$+\frac{1}{\mu}\frac{dp}{dz} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dv(r)}{dr}\right)$$

$\frac{dp}{dz}$ = constant.

steady, fully-developed flow

solve for velocity profile in the pipe, $v(r)$

tangential velocity is neglected here

$$\iint +\frac{r}{\mu}\frac{dp}{dz} = \iint \frac{d}{dr}\left(r\frac{dv(r)}{dr}\right)$$

$$\int \frac{\left(+\frac{r^2}{2\mu}\frac{dp}{dz} + c\right)}{r} = \int \frac{dv(r)}{dr}$$

$$+\frac{r^2}{4\mu}\frac{dp}{dz} + c\ln r + D = v(r)$$

$\Rightarrow$ $v(0)$ is finite so $c = 0$

for no-slip, $v(R) = 0$, so $D = -\frac{R^2}{4\mu}\frac{dp}{dz}$

no slip, incompressible

$$\Rightarrow \quad v_{n,i} = -\frac{1}{4\mu}\frac{dp}{dz}\left(R^2 - r^2\right)$$

flow rate, $q = \int_A v(r)\,dA = 2\pi\int_0^R r v\,dr = \frac{-dp}{dz}\left(\frac{\pi R^4}{8\mu}\right)$

$\Rightarrow$ pressure gradient $\frac{dp}{dz} = \frac{-8\mu q}{\pi R^4}$

$q = \bar{V} A \propto \bar{V} R^2$, so $\frac{dp}{dz} \propto \frac{\bar{V}}{R^2}$

at identical average v, pressure drop grows as $\frac{1}{R^2}$

now consider slip, $v(R) = -s \left.\frac{dv}{dr}\right|_R$

slip coefficient

$$v(r) = \frac{1}{4\mu}\frac{dp}{dz} r^2 + c$$

$$v(R) = \frac{R^2}{4\mu}\frac{dp}{dz} + c = -s\left(\frac{R}{2\mu}\frac{dp}{dz}\right)$$

$$\Rightarrow \quad c = -\frac{1}{\mu}\left(\frac{dp}{dz}\right)\left(\frac{R^2}{4} + \frac{sR}{2}\right)$$

slip contribution

$$\Rightarrow \quad v_{s,i} = -\frac{1}{4\mu}\left(\frac{dp}{dz}\right)\left(R^2 - r^2 + 2sR\right)$$

$$Q = \int_A v(r)\,dA = 2\pi\int_0^R r v\,dr = \left(-\frac{1}{4\mu}\frac{dp}{dz}\right)\left(-\frac{R^4}{4}+\frac{R^4}{2}+sR^3\right)(2\pi)$$

$$Q = -\pi\frac{dp}{dz}\left(\frac{R^4}{8\mu}+\frac{sR^3}{2\mu}\right)$$

$$\frac{dp}{dz} = \frac{-Q}{\frac{\pi}{\mu}\left(\frac{R^4}{8}+\frac{sR^3}{2}\right)}$$

if Q is identical, $\frac{dp}{dz}$ is lower with slip

as $s \to \infty$, $\frac{dp}{dz} \to 0$.

For liquids, $s = b$ = slip length.

For gases, $s = \frac{\mu}{\varepsilon}$ = ratio of internal friction to wall friction

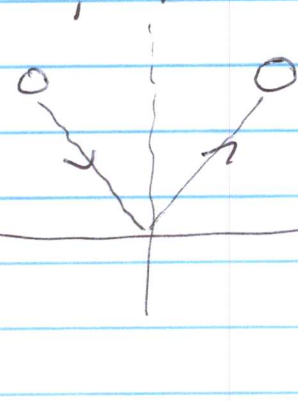
$$\frac{\mu}{\varepsilon} = \frac{\mu}{p}\sqrt{\frac{\pi}{2R_u T_w}}\left(\frac{2-\sigma_v}{\sigma_v}\right)$$

T_w: temp of wall

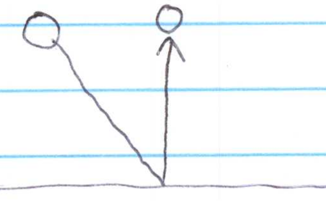
R_u: universal gas constant

$\sigma_v = \frac{\tau_i - \tau_r}{\tau_i}$ = tangential momentum accommodation coefficient, i = incident, r = reflected.

$\sigma_v = 0$, specular.



$\sigma_v = 1$, diffuse



Most gases, $\sigma_v = 0.75 - 0.85$, (Arkilic et al)

see figs 5.4, 5.5, 5.7 for quantification of these effects

Karniadakis + Beskok propose a "unified flow model" for $Kn > 0.1$, such that

$$v(R) = \left(\frac{2-\sigma_v}{\sigma_v}\right)\left(\frac{Kn}{1-bKn}\, 2R \left.\frac{dv}{dr}\right|_R\right)$$

$$\mu_c = \mu\left(\frac{1}{1+Kn}\right)$$ → corrected viscosity

in both slip and unified models, we can also consider gas compressibility, by substituting the ideal gas law.

$$\rho(z) = \frac{p(z)}{R_g T}$$

R_g → specific gas constant

and instead of volume, we must conserve mass flow.

$$\dot{M} = 2\pi\rho(z)\int_0^R r\,v(r)\,dr,$$ if no-slip case then

⟹ differential equation, $$p(z)\frac{dp}{dz} = \frac{-8\mu R_g T}{\pi R^4}$$

and can show eventually that

$$\frac{\dot{M}_{n,t}}{\dot{M}_{n,i}} = \frac{1}{2}\left(\frac{P_i}{P_o} + 1\right)$$

inlet pressure

outlet pressure

no-slip, ratio of mass flows for identical P_i, P_o